Weyl-conformally-invariant lightlike *p*-brane theories: New aspects in black hole physics and Kaluza-Klein dynamics

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We introduce and study in some detail the properties of a novel class of Weyl-conformally invariant *p*-brane theories which describe intrinsically lightlike branes for any odd world-volume dimension. Their dynamics significantly differs from that of the ordinary (conformally noninvariant) Nambu-Goto *p* branes. We present explicit solutions of the Weyl-invariant lightlike brane- (WILL-brane) equations of motion in various gravitational models of physical relevance exhibiting various new phenomena. In $D = 4$ the WILL membrane serves as a material and charged source for gravity and electromagnetism in the coupled Einstein-Maxwell-WILL-membrane system; it automatically positions itself on (straddles) the common event horizon of the corresponding matching black hole solutions, thus providing an explicit dynamical realization of the membrane paradigm in black hole physics. In product spaces of interest in Kaluza-Klein theories the WILL-brane wraps nontrivially around the compact (internal) dimensions and still describes massless mode dynamics in the noncompact (space-time) dimensions. Because of nontrivial variable size of the internal compact dimensions we find new types of physically interesting solutions describing massless brane modes trapped on bounded planar circular orbits with nontrivial angular momentum, and with linear dependence between energy and angular momentum.

I. INTRODUCTION

In the recent years there has been a considerable interest in the study of higher-dimensional extended objects motivated by various developments in string theory, gravity, astrophysics and cosmology.

In nonperturbative string theory there arise several types of higher-dimensional membranes (*p* branes, *Dp* branes) which play a crucial role in the description of string dualities, microscopic physics of black holes, gauge theory/gravity correspondence, large-radius compactifications of extra dimensions, cosmological brane-world scenarios in high-energy particle phenomenology, etc. (for a background on string and brane theories, see Refs. [1]).

In the context of black hole physics, the so called "membrane paradigm" [2] appears to be a quite effective treatment of the physics of a black hole horizon. Furthermore, the thin-wall description of domain walls coupled to gravity [3,4] is able to provide neat models for many cosmological and astrophysical effects.

It seems therefore of fundamental importance that all kinds of higher-dimensional extended objects, which could be consistently formulated, and their possible role in the various areas of physics should be thoroughly investigated.

Lightlike membranes are indeed of great importance in general relativity and have been extensively studied from a phenomenological point of view [3,4], i.e., by introducing them without specifying the Lagrangian dynamics from which they may originate. These lightlike membranes have been treated as a source of gravity enabling the formulation of important effects in the context of black hole physics.

In the present paper we develop in some detail a new field-theoretic approach for a systematic description of the dynamics of lightlike *p* branes starting from a concise Weyl-conformally invariant action. Part of the results have been previously reported in shorter form in Refs. [5]. Our approach is based on the general idea of employing alternative non-Riemannian integration measures (volume forms) in the actions of generally-covariant (reparametrization-invariant) field theories instead of (or, more generally, on equal footing with) the standard Riemannian volume forms. This idea has been first proposed and applied in the context of four-dimensional theories involving gravity [6] by introducing a new class of "two-measure" gravitational models. It has been demonstrated that the latter models are capable to provide plausible solutions for a broad array of basic problems in cosmology and particle physics, such as: (i) scale invariance and its dynamical breakdown; (ii) spontaneous generation of dimensionful fundamental scales; (iii) the cosmological-constant problem; (iv) the problem of fermionic families; (v) applications to dark energy problem and modern cosmological brane-world scenarios.

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For a detailed discussion we refer to the series of papers [6,7].

Subsequently, the idea of employing an alternative non-Riemannian integration measure was applied systematically to string, p-brane and Dp-brane models [8]. The main feature of these new classes of modified string/brane theories is the appearance of the pertinent string/brane tension as an additional dynamical degree of freedom beyond the usual string/brane physical degrees of freedom, instead of being introduced *ad hoc* as a dimensionful scale. The dynamical string/brane tension acquires the physical meaning of a world-sheet electric field strength (in the string case) or world-volume $(p+1)$-form field strength (in the p-brane case) and obeys Maxwell (Yang-Mills) equations of motion or their higher-rank antisymmetric tensor gauge field analogues, respectively. As a result of the latter property the modified-measure string model with dynamical tension yields a simple classical mechanism of "color" charge confinement [8].

One drawback of modified-measure p-brane and Dp-brane models, similarly to a drawback of ordinary Nambu-Goto p-branes, is that Weyl-conformal invariance is lost beyond the simplest string case ($p = 1$). On the other hand, it turns out that the form of the action of the modified-measure string model with dynamical tension suggests a natural way to construct explicitly a substantially new class of *Weyl-conformally invariant p*-brane models *for any p* [5]. The most profound property of the latter models is that for any even p they describe the dynamics of inherently *lightlike p* branes which makes them significantly different both from the standard Nambu-Goto (or Dirac-Born-Infeld) branes as well as from their modified versions with dynamical string/brane tensions [8] mentioned above.

Let us note that various papers have previously appeared in the literature [9] where the standard Weyl-conformally noninvariant Nambu-Goto p-brane action (Eq. (8) below) and its supersymmetric counterparts were reformulated in a formally Weyl-invariant form by means of introducing auxiliary nondynamical fields with a nontrivial transformation properties under Weyl-conformal symmetry appropriately tuned up to compensate for the Weyl noninvariance with respect to (w.r.t.) the original dynamical degrees of freedom. However, one immediately observes that the latter formally Weyl-invariant p-brane actions *do not change* the dynamical content of the standard Nambu-Goto p-branes (describing inherently *massive* modes). This is in sharp contrast to the new Weyl-conformally invariant p-brane models introduced and studied in detail below, which describe *intrinsically lightlike p* branes for any even p. In what follows we will use for the latter the acronym Weyl-invariant lightlike-(WILL)-branes.

In the present paper we will demonstrate that WILL branes can play a very interesting role in diverse areas of physics. We begin with a short review of the concept of alternative non-Riemannian volume form (integration measure) in the context of string and p-brane models (Section II). In Sec. III, after a brief reminder of the standard Polyakov-type formulation of ordinary Nambu-Goto p-branes, we introduce and describe the Lagrangian formulation of the new class of inherently Weyl-invariant p branes for any p and exhibit their intrinsic lightlike nature when p is even (WILL branes). In Sec. IV and forward we study in detail the properties of WILL membranes (i.e., for $p = 2$), in particular, we introduce a natural coupling of the WILL membrane to external space-time electromagnetic fields.

The role of WILL membranes in the context of gravity is discussed in Secs. V and VI. When moving as a test brane in a $D = 4$ black hole gravitational background the WILL membrane ($p = 2$) automatically locates itself on the event horizon (Sec. V). Furthermore, as shown in Sec. VI, the WILL membrane can serve as a material and charged source for gravity and electromagnetism in the coupled Einstein-Maxwell-WILL-membrane system. We derive a self-consistent solution where the WILL membrane locates itself on ("straddles") the common event horizon of two black holes (Reissner-Nordström in the exterior and Schwarzschild in the interior). Therefore, the WILL membrane provides an explicit dynamical realization of the membrane paradigm in black hole physics [2].

The role of WILL membranes in the context of Kaluza-Klein theories is studied in Section VII where we consider WILL-membrane dynamics in higher-dimensional product-type space-time. It is shown that the WILL membrane describes massless particlelike modes while acquiring nontrivial Kaluza-Klein quantum numbers. When the size of extra compact dimensions is constant the motion of these massless brane modes is indistinguishable from that of ordinary massless point particles w.r.t. the projected $D = 4$ world. An interesting new feature arises when the size of the extra compact dimensions has nontrivial space dependence. In this case we find an explicit solution describing massless particlelike brane mode motion on the noncompact $D = 4$ space-time, where the modes are trapped on bounded planar circular orbits with a linear relation between energy and angular momentum, while winding nontrivially the extra compact dimensions. The latter feature is inaccessible in standard Kaluza-Klein models.

The last section collects some conclusions and outlook for future studies of the role and further aspects of WILL-brane dynamics.

II. STRING AND BRANE MODELS WITH A MODIFIED WORLD-SHEET/WORLD-VOLUME INTEGRATION MEASURE

The modified-measure bosonic string model is given by the following action [8]:

$$S = -\int d^2\sigma \Phi(\varphi)\left[\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)\right.$$
$$\left. -\frac{\varepsilon^{ab}}{2\sqrt{-\gamma}}F_{ab}(A)\right] + \int d^2\sigma\sqrt{-\gamma}A_a J^a, \quad (1)$$

$$J^a = \frac{\varepsilon^{ab}}{\sqrt{-\gamma}}\partial_b u,$$

with the notations:

$$\Phi(\varphi) \equiv \frac{1}{2}\varepsilon_{ij}\varepsilon^{ab}\partial_a\varphi^i\partial_b\varphi^j, \qquad F_{ab}(A) = \partial_a A_b - \partial_b A_a.$$
$$(2)$$

Here φ^i denote auxiliary world-sheet scalar fields, γ_{ab} indicates the intrinsic Riemannian world-sheet metric with $\gamma = \det\|\gamma_{ab}\|$ and $G_{\mu\nu}(X)$ is the Riemannian metric of the embedding space-time ($a, b = 0, 1; i, j = 1, 2; \mu, \nu = 0, 1, \ldots, D-1$).

In action [8] we notice the following differences w.r.t. the standard Nambu-Goto string (in the Polyakov-like formulation):

(a) New non-Riemannian integration measure density $\Phi(\varphi)$ instead of $\sqrt{-\gamma}$;
(b) Dynamical string tension $T \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}}$ instead of *ad hoc* dimensionful constant;
(c) Auxiliary world-sheet gauge field A_a in a would-be "topological" term $\int d^2\sigma \frac{\Phi(\varphi)}{\sqrt{-\gamma}}\frac{1}{2}\varepsilon^{ab}F_{ab}(A)$;
(d) Optional natural coupling of auxiliary A_a to external conserved world-sheet electric current J^a (see last equality in [8] and Eq. (4) below).

The modified string model [8] is Weyl-conformally invariant similarly to the standard Polyakov formulation. Here Weyl-conformal symmetry is given by Weyl rescaling of γ_{ab} supplemented with a special diffeomorphism in φ-target space:

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \rho\gamma_{ab},$$
$$\varphi^i \longrightarrow \varphi'^i = \varphi'^i(\varphi) \quad \text{with} \quad \det\left\|\frac{\partial\varphi'^i}{\partial\varphi^j}\right\| = \rho. \quad (3)$$

The dynamical string tension appears as a canonically conjugated momentum w.r.t. A_1: $\pi_{A_1} \equiv \frac{\partial\mathcal{L}}{\partial\dot{A}_1} = \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \equiv T$, i.e., T has the meaning of a *world-sheet electric field strength*, and the equations of motion w.r.t. auxiliary gauge field A_a look exactly as $D = 2$ Maxwell equations:

$$\frac{\varepsilon^{ab}}{\sqrt{-\gamma}}\partial_b T + J^a = 0. \quad (4)$$

In particular, for $J^a = 0$:

$$\varepsilon^{ab}\partial_b\left(\frac{\Phi(\varphi)}{\sqrt{-\gamma}}\right) = 0, \qquad \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \equiv T = \text{const.}, \quad (5)$$

one gets a *spontaneously induced* constant string tension. Furthermore, when the modified string couples to pointlike

charges on the world-sheet (i.e., $J^0\sqrt{-\gamma} = \sum_i e_i\delta(\sigma - \sigma_i)$ in (4)) one obtains classical charge *confinement*: $\sum_i e_i = 0$.

The above charge confinement mechanism has also been generalized in [8] to the case of coupling the modified string model (with dynamical tension) to non-Abelian world-sheet color charges. The latter is achieved as follows. Notice the following identity in 2D involving Abelian gauge field A_a:

$$\frac{\varepsilon^{ab}}{2\sqrt{-\gamma}}F_{ab}(A) = \sqrt{-\frac{1}{2}F_{ab}(A)F_{cd}(A)\gamma^{ac}\gamma^{bd}}. \quad (6)$$

Using (6) the extension of the action [8] to the non-Abelian case is straightforward:

$$S = -\int d^2\sigma\Phi(\varphi)\left[\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)\right.$$
$$\left. -\sqrt{-\frac{1}{2}\text{Tr}(F_{ab}(A)F_{cd}(A))\gamma^{ac}\gamma^{bd}}\right]$$
$$+ \int d^2\sigma\text{Tr}(A_a j^a), \quad (7)$$

with $F_{ab}(A) = \partial_a A_b - \partial_b A_c + i[A_a, A_b]$. The model (7) shares the same principal properties as the model (1)—dynamical generation of string tension as an additional degree of freedom, non-Abelian color charge confinement already on the classical level, etc.

Similar construction has also been proposed in [8] for higher-dimensional modified-measure p- and Dp-brane models whose brane tension appears as an additional dynamical degree of freedom. On the other hand, like the standard Nambu-Goto branes, they are Weyl-conformally *non*invariant and describe dynamics of *massive* modes.

III. WEYL-INVARIANT BRANES

A. Standard Nambu-Goto branes

Before proceeding to the main exposition, let us briefly recall the standard Polyakov-type formulation of the ordinary (bosonic) Nambu-Goto p-brane action:

$$S = -\frac{T}{2}\int d^{p+1}\sigma\sqrt{-\gamma}[\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)$$
$$- \Lambda(p-1)]. \quad (8)$$

Here γ_{ab} is the ordinary Riemannian metric on the $p + 1$-dimensional brane world volume with $\gamma \equiv \det\|\gamma_{ab}\|$. The world-volume indices $a, b = 0, 1, \ldots, p$; $G_{\mu\nu}$ denotes the Riemannian metric in the embedding space-time with space-time indices $\mu, \nu = 0, 1, \ldots, D-1$. T is the given *ad hoc* brane tension; the constant Λ can be absorbed by rescaling T (see below Eq. (14)). The equations of motion w.r.t. γ^{ab} and X^μ read

$$T_{ab} \equiv \left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X^\nu \right) G_{\mu\nu}$$
$$+ \gamma_{ab} \frac{\Lambda}{2} (p - 1) = 0, \quad (9)$$

$$\partial_a (\sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu) + \sqrt{-\gamma} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0, \quad (10)$$

where

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} G^{\mu\kappa} (\partial_\nu G_{\kappa\lambda} + \partial_\lambda G_{\kappa\nu} - \partial_\kappa G_{\nu\lambda}) \quad (11)$$

is the Christoffel connection for the external metric.

Equations (9) when $p \neq 1$ imply

$$\Lambda \gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \quad (12)$$

which in turn allows to rewrite Eq. (9) as

$$T_{ab} \equiv \left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{p+1} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X^\nu \right) G_{\mu\nu} = 0.$$
$$(13)$$

Furthermore, using (12) the Polyakov-type brane action (8) becomes on-shell equivalent to the Nambu-Goto-type brane action:

$$S = -T \Lambda^{-[(p-1)/2]} \int d^{p+1}\sigma \sqrt{-\det||\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}||}.$$
$$(14)$$

Let us note the following properties of standard Nambu-Goto p branes manifesting their crucial differences w.r.t. the Weyl-conformally invariant branes discussed below. Equation (12) tells us that: (i) the induced metric on the Nambu-Goto p-brane world volume is *nonsingular*; (ii) standard Nambu-Goto p branes describe intrinsically *massive* modes.

B. Weyl-invariant branes: action and equations of motion

Identity (6) and the modified-measure string action (7) naturally suggest how to construct Weyl-invariant p-brane models for any p. Namely, we consider the following novel class of p-brane actions:

$$S = - \int d^{p+1}\sigma \Phi(\varphi) \left[\frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \right.$$
$$\left. - \sqrt{F_{ab}(A) F_{cd}(A) \gamma^{ac} \gamma^{bd}} \right], \quad (15)$$

$$\Phi(\varphi) \equiv \frac{1}{(p+1)!} \varepsilon_{i_1 \ldots i_{p+1}} \varepsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} \varphi^{i_1} \ldots \partial_{a_{p+1}} \varphi^{i_{p+1}},$$
$$(16)$$

where notations similar to those in [8] are used (here $a, b = 0, 1, \ldots, p; i, j = 1, \ldots, p + 1$). In particular, φ^i are world-volume scalar fields which are the building blocks of the non-Riemannian integration measure (16).

The above action (15) is invariant under Weyl-conformal symmetry (the same as in the dynamical-tension string model [8]):

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \rho \gamma_{ab},$$
$$\varphi^i \longrightarrow \varphi'^i = \varphi'^i(\varphi) \quad \text{with} \quad \det \left\| \frac{\partial \varphi'^i}{\partial \varphi^j} \right\| = \rho. \quad (17)$$

Let us note the following significant differences of (15) w.r.t. the standard Nambu-Goto p-branes (in the Polyakov-like formulation):

(a) New non-Riemannian integration measure density $\Phi(\varphi)$ instead of $\sqrt{-\gamma}$, and *no* "cosmological-constant" term $((p - 1)\sqrt{-\gamma})$;

(b) Variable brane tension $\chi \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}}$ which is now Weyl-conformally *gauge dependent*: $\chi \to \rho^{\frac{1}{2}(1-p)} \chi$;

(c) Auxiliary world-volume gauge field A_a in a "-square-root" Maxwell-type term[1]; the latter can be straightforwardly generalized to the non-Abelian case—$\sqrt{-\text{Tr}(F_{ab}(A) F_{cd}(A)) \gamma^{ac} \gamma^{bd}}$ similar to (7);

(d) Natural optional couplings of the auxiliary gauge field A_a to external world-volume color charge currents j^a as in (7);

(e) The action (15) is manifestly Weyl-conformal invariant for *any* p; it describes *intrinsically lightlike p* branes for any even p, as it will be shown below.

In what follows we shall frequently use the shorthand notations:

$$(\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$
$$\sqrt{FF\gamma\gamma} \equiv \sqrt{F_{ab} F_{cd} \gamma^{ac} \gamma^{bd}}. \quad (18)$$

Employing (18) the equations of motion w.r.t. measure-building auxiliary scalars φ^i and w.r.t. γ^{ab} read, respectively:

$$\frac{1}{2} \gamma^{cd} (\partial_c X \partial_d X) - \sqrt{FF\gamma\gamma} = M (= \text{const.}), \quad (19)$$

$$\frac{1}{2} (\partial_a X \partial_b X) + \frac{F_{ac} \gamma^{cd} F_{db}}{\sqrt{FF\gamma\gamma}} = 0. \quad (20)$$

Taking the trace in (20) implies $M = 0$ in Eq. (19).

Next we have the following equations of motion w.r.t. auxiliary gauge field A_a and w.r.t. X^μ, respectively:

$$\partial_b \left(\frac{F_{cd} \gamma^{ac} \gamma^{bd}}{\sqrt{FF\gamma\gamma}} \Phi(\varphi) \right) = 0, \quad (21)$$

$$\partial_a (\Phi(\varphi) \gamma^{ab} \partial_b X^\mu) + \Phi(\varphi) \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0, \quad (22)$$

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[1]"Square-root" Maxwell (Yang-Mills) action in $D = 4$ was originally introduced in the first Ref. [10] and later generalized to "square-root" actions of higher-rank antisymmetric tensor gauge fields in $D \geq 4$ in the second and third Refs. [10].

where $\Gamma^\mu_{\nu\lambda}$ is the Christoffel connection corresponding to the external space-time metric $G_{\mu\nu}$ as in (11).

C. Intrinsically lightlike branes

Let us consider the γ^{ab}-equations of motion (20). F_{ab} is an antisymmetric $(p+1) \times (p+1)$ matrix, therefore, F_{ab} is *not invertible* in any odd $(p+1)$—it has at least one zero-eigenvalue vector V^a ($F_{ab}V^b = 0$). Therefore, for any odd $(p+1)$ the induced metric:

$$g_{ab} \equiv (\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \qquad (23)$$

on the world volume of the Weyl-invariant brane (15) is *singular* as *opposed* to the ordinary Nambu-Goto brane (where the induced metric is proportional to the intrinsic Riemannian world-volume metric, cf. Equation (12)):

$$(\partial_a X \partial_b X) V^b = 0, \qquad \text{i.e.} \quad (\partial_V X \partial_V X) = 0,$$
$$(\partial_\perp X \partial_V X) = 0, \qquad (24)$$

where $\partial_V \equiv V^a \partial_a$ and $\partial_\perp$ are derivatives along the tangent vectors in the complement of the tangent vector field V^a.

Thus, we arrive at the following important conclusion: every point on the world surface of the Weyl-invariant p brane (15) (for odd $(p+1)$) moves with the speed of light in a time evolution along the zero-eigenvalue vector-field V^a of F_{ab}. Therefore, we will name (15) (for odd $(p+1)$) by the acronym WILL-brane model.

D. Dual formulation of WILL branes

The A_a equations of motion (21) can be solved in terms of $(p-2)$-form gauge potentials $\Lambda_{a_1...a_{p-2}}$ dual w.r.t. A_a. The respective field strengths are related as follows:

$$F_{ab}(A) = -\frac{1}{\chi} \frac{\sqrt{-\gamma} \varepsilon_{abc_1...c_{p-1}}}{2(p-1)} \gamma^{c_1 d_1} \dots \gamma^{c_{p-1} d_{p-1}}$$
$$\times F_{d_1...d_{p-1}}(\Lambda) \gamma^{cd}(\partial_c X \partial_d X), \qquad (25)$$

$$\chi^2 = -\frac{2}{(p-1)^2} \gamma^{a_1 b_1} \dots \gamma^{a_{p-1} b_{p-1}} F_{a_1...a_{p-1}}(\Lambda) F_{b_1...b_{p-1}}(\Lambda), \qquad (26)$$

where $\chi \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}}$ is the variable brane tension, and

$$F_{a_1...a_{p-1}}(\Lambda) = (p-1)\partial_{[a_1} \Lambda_{a_2...a_{p-1}]} \qquad (27)$$

is the $(p-1)$-form dual field strength.

All equations of motion (19)–(22) can be equivalently derived from the following *dual* WILL-brane action:

$$S_{\text{dual}}[X, \gamma, \Lambda] = -\frac{1}{2} \int d^{p+1}\sigma \chi(\gamma, \Lambda)$$
$$\times \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \qquad (28)$$

with $\chi(\gamma, \Lambda)$ given in (26) above. In particular, in terms of the dual gauge fields (27), (20), and (21) read

$$(\partial_a X \partial_b X) + \frac{1}{2}\gamma^{cd}(\partial_c X \partial_c X)\left[-\gamma_{ab} + (p-1)\right.$$
$$\left. \times \frac{\gamma^{a_1 b_1} \dots \gamma^{a_{p-2}b_{p-2}} F_{aa_1...a_{p-2}}(\Lambda) F_{bb_1...b_{p-2}}(\Lambda)}{\gamma^{c_1 d_1} \dots \gamma^{c_{p-1}d_{p-1}} F_{c_1...c_{p-1}}(\Lambda) F_{d_1...d_{p-1}}(\Lambda)}\right] = 0, \quad (29)$$

$$\partial_b\left(\gamma^{a_1 b_1} \dots \gamma^{a_{p-2}b_{p-2}} \gamma^{bb_{p-1}} F_{b_1...b_{p-1}}(\Lambda) \frac{1}{\chi(\gamma, \Lambda)}\right.$$
$$\left. \times \sqrt{-\gamma} \gamma^{cd}(\partial_c X \partial_c X)\right) = 0, \quad (30)$$

with $\chi(\gamma, \Lambda)$ as in (26).

IV. SPECIAL CASE $p = 2$. COUPLING TO EXTERNAL ELECTROMAGNETIC FIELD

A. WILL membrane

Henceforth we will explicitly consider the special case $p = 2$ of (15) i.e., the Weyl-invariant lightlike membrane model:

$$S = -\int d^3\sigma \Phi(\varphi)\left[\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)\right.$$
$$\left. -\sqrt{F_{ab}(A)F_{cd}(A)\gamma^{ac}\gamma^{bd}}\right], \qquad (31)$$

$$\Phi(\varphi) \equiv \frac{1}{3!}\varepsilon_{ijk}\varepsilon^{abc}\partial_a\varphi^i \partial_b\varphi^j \partial_c\varphi^k,$$
$$a, b, c = 0, 1, 2, \qquad i, j, k = 1, 2, 3. \qquad (32)$$

The associated WILL-membrane dual action (particular case of (28) for $p = 2$) reads

$$S_{\text{dual}} = -\frac{1}{2}\int d^3\sigma \chi(\gamma, u)\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X),$$
$$\chi(\gamma, u) \equiv \sqrt{-2\gamma^{cd}\partial_c u \partial_d u}, \qquad (33)$$

where u is the dual "gauge" potential w.r.t. A_a:

$$F_{ab}(A) = -\frac{1}{2\chi(\gamma, u)}\sqrt{-\gamma}\varepsilon_{abc}\gamma^{cd}\partial_d u \gamma^{ef}(\partial_e X \partial_f X). \qquad (34)$$

S_{dual} is manifestly Weyl invariant (under $\gamma_{ab} \to \rho\gamma_{ab}$).

The equations of motion w.r.t. γ^{ab}, u (or A_a), and X^μ read accordingly (using again shorthand notation (23)):

$$(\partial_a X \partial_b X) + \frac{1}{2}\gamma^{cd}(\partial_c X \partial_d X)\left(\frac{\partial_a u \partial_b u}{\gamma^{ef}\partial_e u \partial_f u} - \gamma_{ab}\right) = 0, \qquad (35)$$

$$\partial_a\left(\frac{\sqrt{-\gamma}\gamma^{ab}\partial_b u}{\chi(\gamma, u)}\gamma^{cd}(\partial_c X \partial_d X)\right) = 0, \qquad (36)$$

$$\partial_a(\chi(\gamma, u)\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) + \chi(\gamma, u)$$
$$\times \sqrt{-\gamma}\gamma^{ab}\partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0. \quad (37)$$

Equation (35) represents the relation between the intrinsic metric γ_{ab} and the induced metric $(\partial_a X \partial_b X)$. However, the last factor in brackets on the l.h.s. of (35) is a projector implying that the induced metric $g_{ab} \equiv (\partial_a X \partial_b X)$ has zero-mode eigenvector $V^a = \gamma^{ab}\partial_b u$.

B. Gauge-fixed constraints and equations of motion

The invariance under world-volume reparametrizations allows to introduce the following standard (synchronous) gauge-fixing conditions:

$$\gamma^{0i} = 0 \ (i = 1, 2), \qquad \gamma^{00} = -1. \quad (38)$$

Using (38) we can easily find solutions of Eq. (36) for the dual "gauge potential" u in spite of its high nonlinearity by taking the following ansatz:

$$u(\tau, \sigma^1, \sigma^2) = \frac{T_0}{\sqrt{2}}\tau. \quad (39)$$

Here T_0 is an arbitrary integration constant with the dimension of membrane tension. In particular,

$$\chi \equiv \sqrt{-2\gamma^{ab}\partial_a u \partial_b u} = T_0. \quad (40)$$

The ansatz (39) means that we take $\tau \equiv \sigma^0$ to be evolution parameter along the zero-eigenvalue vector field of the induced metric on the brane $(V^a = \gamma^{ab}\partial_b u = \text{const.}(1, 0, 0))$. Also, in terms of the original gauge field A_a (cf. relation (34)) Eq. (39) implies vanishing of the world-volume "electric" field strength $F_{0i}(A) = 0$.

The ansatz for u (39) together with the gauge choice for γ_{ab} (38) brings the equations of motion w.r.t. γ^{ab}, u (or A_a) and X^μ in the following form (recall $(\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$):

$$(\partial_0 X \partial_0 X) = 0, \qquad (\partial_0 X \partial_i X) = 0, \quad (41)$$

$$(\partial_i X \partial_j X) - \frac{1}{2}\gamma_{ij}\gamma^{kl}(\partial_k X \partial_l X) = 0, \quad (42)$$

(notice that Eqs. (42) look exactly like the classical (Virasoro) constraints for an Euclidean string theory with world-sheet parameters (σ^1, σ^2));

$$\partial_0(\sqrt{\gamma_{(2)}}\gamma^{kl}(\partial_k X \partial_l X)) = 0, \quad (43)$$

where $\gamma_{(2)} = \det\|\gamma_{ij}\|$;

$$\Box^{(3)}X^\mu + (-\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{kl}\partial_k X^\nu \partial_l X^\lambda)\Gamma^\mu_{\nu\lambda} = 0, \quad (44)$$

where

$$\Box^{(3)} \equiv -\frac{1}{\sqrt{\gamma^{(2)}}}\partial_0(\sqrt{\gamma^{(2)}}\partial_0) + \frac{1}{\sqrt{\gamma^{(2)}}}\partial_i(\sqrt{\gamma^{(2)}}\gamma^{ij}\partial_j).$$
$$(45)$$

Let us note that Eq. (43) is the only remnant from the A_a-equations of motion (21) and, in fact, it can easily be shown that (43) is a consequence of the gauge-fixed constraints (41) and (42) and equations of motion (44).

C. Coupling to external electromagnetic field

We can also extend the WILL-brane model (15) via a coupling to external space-time electromagnetic field $\mathcal{A}_\mu$. The natural Weyl-conformal invariant candidate action reads (for $p = 2$):

$$S_{\text{WILL-brane}} = -\int d^3\sigma \Phi(\varphi)\left[\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}\right.$$
$$\left. - \sqrt{F_{ab}F_{cd}\gamma^{ac}\gamma^{bd}}\right]$$
$$+ q\int d^3\sigma \varepsilon^{abc}\mathcal{A}_\mu \partial_a X^\mu F_{bc}. \quad (46)$$

The last Chern-Simons-like term is a special case of a class of Chern-Simons-like couplings of extended objects to external electromagnetic fields proposed in Ref. [11].

Instead of the action (46) we can use its dual one (similar to the simpler case Eq. (15) versus Eq. (33)):

$$S^{\text{dual}}_{\text{WILL-brane}} = -\frac{1}{2}\int d^3\sigma \chi(\gamma, u, \mathcal{A})\sqrt{-\gamma}\gamma^{ab}(\partial_a X \partial_b X),$$
$$(47)$$

where the variable brane tension $\chi \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}}$ is given by

$$\chi(\gamma, u, \mathcal{A}) \equiv \sqrt{-2\gamma^{cd}(\partial_c u - q\mathcal{A}_c)(\partial_d u - q\mathcal{A}_d)},$$
$$\mathcal{A}_a \equiv \mathcal{A}_\mu \partial_a X^\mu. \quad (48)$$

Here u is the dual gauge potential w.r.t. A_a and the corresponding field strength and dual field strength are related as (cf. Equation (34)):

$$F_{ab}(A) = -\frac{1}{2\chi(\gamma, u, \mathcal{A})}$$
$$\times \sqrt{-\gamma}\varepsilon_{abc}\gamma^{cd}(\partial_d u - \mathcal{A}_d)\gamma^{ef}(\partial_e X \partial_f X). \quad (49)$$

The corresponding equations of motion w.r.t. γ^{ab}, u (or A_a), and X^μ read accordingly:

$$(\partial_a X \partial_b X) + \frac{1}{2}\gamma^{cd}(\partial_c X \partial_d X)$$
$$\times \left(\frac{(\partial_a u - q\mathcal{A}_a)(\partial_b u - q\mathcal{A}_b)}{\gamma^{ef}(\partial_e u - q\mathcal{A}_e)(\partial_f u - q\mathcal{A}_f)} - \gamma_{ab}\right) = 0; \quad (50)$$

$$\partial_a\left(\frac{\sqrt{-\gamma}\gamma^{ab}(\partial_b u - q\mathcal{A}_b)}{\chi(\gamma, u, \mathcal{A})}\gamma^{cd}(\partial_c X \partial_d X)\right) = 0; \quad (51)$$

$$\partial_a(\chi(\gamma, u, \mathcal{A})\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) + \chi(\gamma, u, \mathcal{A})$$
$$\times \sqrt{-\gamma}\gamma^{ab}\partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda}$$
$$+ q\varepsilon^{abc}F_{bc}\partial_a X^\nu(\partial_\lambda \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\lambda)G^{\lambda\mu} = 0. \quad (52)$$

V. WILL-MEMBRANE SOLUTIONS IN VARIOUS GRAVITATIONAL BACKGROUNDS

A. Example: WILL membrane in a pp-wave background

As a first nontrivial example let us consider WILL-membrane dynamics in an external background generalizing the plane-polarized gravitational wave (pp wave):

$$(ds)^2 = -dx^+ dx^- - F(x^+, x^I)(dx^+)^2 + h_{IJ}(x^K)dx^I dx^J, \quad (53)$$

(for the ordinary pp wave $h_{IJ}(x^K) = \delta_{IJ}$), and let us employ in (41)–(45) the following natural ansatz for X^μ (here $\sigma^0 \equiv \tau$; $I = 1, \ldots, D - 2$):

$$X^- = \tau, \qquad X^+ = X^+(\tau, \sigma^1, \sigma^2), \qquad X^I = X^I(\sigma^1, \sigma^2). \quad (54)$$

The nonzero affine connection symbols for the generalized pp-wave metric (53) are: $\Gamma^-_{++} = \partial_+ F$, $\Gamma^-_{+I} = \partial_I F$, $\Gamma^I_{++} = \frac{1}{2}h^{IJ}\partial_J F$, and Γ^I_{JK}—the ordinary Christoffel symbols for the metric h_{IJ} in the transverse dimensions.

It is straightforward to show that the solution does not depend on the form of the pp-wave front $F(x^+, x^I)$ and reads

$$X^+ = X^+_0 = \text{const.}, \qquad \gamma_{ij} \quad \text{are} \quad \tau - \text{independent}; \quad (55)$$

$$\left(\partial_i X^I \partial_j X^J - \frac{1}{2}\gamma_{ij}\gamma^{kl}\partial_k X^I \partial_l X^J\right)h_{IJ} = 0, \quad (56)$$

$$\frac{1}{\sqrt{\gamma^{(2)}}}\partial_i\left(\sqrt{\gamma^{(2)}}\gamma^{ij}\partial_j X^I\right) + \gamma^{kl}\partial_k X^J \partial_l X^K \Gamma^I_{JK} = 0. \quad (57)$$

The latter two equations for the transverse brane coordinates describe a string moving in the $(D - 2)$-dimensional Euclidean-signature transverse space.

B. Example: WILL membrane in spherically-symmetric backgrounds

Let us consider general SO(3)-symmetric background in $D = 4$ embedding space-time:

$$(ds)^2 = -A(z, t)(dt)^2 + B(z, t)(dz)^2$$
$$+ C(z, t)((d\theta)^2 + \sin^2\theta(d\phi)^2). \quad (58)$$

The usual ansatz:

$$X^0 \equiv t = \tau, \qquad X^1 \equiv z = z(\tau, \sigma^1, \sigma^2),$$
$$X^2 \equiv \theta = \sigma^1, \qquad X^3 \equiv \phi = \sigma^2, \quad (59)$$
$$\gamma_{ij} = a(\tau)((d\sigma^1)^2 + \sin^2(\sigma^1)(d\sigma^2)^2)$$

yields:
(i) equations for $z(\tau, \sigma^1, \sigma^2)$:

$$\frac{\partial z}{\partial \tau} = \pm\sqrt{\frac{A}{B}}, \qquad \frac{\partial z}{\partial \sigma^i} = 0; \quad (60)$$

(ii) a restriction on the background itself (comes from the gauge-fixed equations of motion for the dual gauge potential u (43)) :

$$\frac{dC}{d\tau} \equiv \left(\frac{\partial C}{\partial t} \pm \sqrt{\frac{A}{B}}\frac{\partial C}{\partial z}\right)\bigg|_{t=\tau, z=z(\tau)} = 0; \quad (61)$$

(iii) an equation for the conformal factor $a(\tau)$ of the internal membrane metric:

$$\partial_\tau a + \left(\frac{\frac{\partial}{\partial t}\sqrt{AB} \pm \partial_z A}{\sqrt{AB}}\bigg|_{t=\tau, z=z(\tau)}\right)a(\tau) - \frac{\frac{\partial}{\partial t}C}{A}\bigg|_{t=\tau, z=z(\tau)} = 0. \quad (62)$$

Equation (61) tells that the (squared) sphere radius $R^2 \equiv C(z, t)$ must remain constant along the WILL-brane trajectory.

C. Example: WILL membrane in Schwarzschild and Reissner-Nordström black holes

Let us apply the results of Sec. V B for static spherically-symmetric gravitational background in $D = 4$:

$$(ds)^2 = -A(r)(dt)^2 + B(r)(dr)^2$$
$$+ r^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2]. \quad (63)$$

Specifically we have

$$A(r) = B^{-1}(r) = 1 - \frac{2GM}{r} \quad (64)$$

for Schwarzschild black hole,

$$A(r) = B^{-1}(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} \quad (65)$$

for Reissner-Nordström black hole,

$$A(r) = B^{-1}(r) = 1 - \kappa r^2 \quad (66)$$

for (anti-) de Sitter space, etc.
In the case of (63), Eqs. (60) and (61) reduce to

$$\frac{\partial r}{\partial \tau} = \pm A(r), \qquad \frac{\partial r}{\partial \sigma^i} = 0, \qquad \frac{\partial r}{\partial \tau} = 0 \quad (67)$$

yielding:

$$r = r_0 \equiv \text{const.}, \qquad \text{where} \quad A(r_0) = 0. \qquad (68)$$

Further, Eq. (62) implies for the intrinsic WILL-membrane metric:

$$\|\gamma_{ij}\| = c_0 e^{\mp \tau/r_0} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\sigma^1) \end{pmatrix}, \qquad (69)$$

where c_0 is an arbitrary integration constant.

From (68) we conclude that the WILL membrane with spherical topology (and with exponentially blowing-up/deflating radius w.r.t. internal metric, see Eq. (69)) automatically "sits" on (straddles) the event horizon of the pertinent black hole in $D = 4$ embedding space-time. This conforms with the well-known general property of closed lightlike hypersurfaces in $D = 4$ (i.e., their section with the hyper-plane $t = $ const. being a compact 2-dimensional manifold) which automatically serve as horizons [12]. On the other hand, let us stress that our WILL-membrane model (33) provides an explicit *dynamical* realization of event horizons.

VI. COUPLED EINSTEIN-MAXWELL-WILL-MEMBRANE SYSTEM: WILL MEMBRANE AS A SOURCE FOR GRAVITY AND ELECTROMAGNETISM

We can extend the results from the previous section to the case of the self-consistent Einstein-Maxwell-WILL-membrane system, i.e., we will consider the WILL membrane as a dynamical material and electrically charged source for gravity and electromagnetism. The relevant action reads

$$S = \int d^4 x \sqrt{-G} \left[\frac{R(G)}{16\pi G_N} \right.$$
$$\left. - \frac{1}{4} \mathcal{F}_{\mu\nu}(\mathcal{A}) \mathcal{F}_{\kappa\lambda}(\mathcal{A}) G^{\mu\kappa} G^{\nu\lambda} \right] + S_{\text{WILL-brane}},$$
$$(70)$$

where $\mathcal{F}_{\mu\nu}(\mathcal{A}) = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the space-time electromagnetic field strength, and $S_{\text{WILL-brane}}$ indicates the WILL-membrane action coupled to the space-time gauge field $\mathcal{A}_\mu$—either (46) or its dual (47).

The equations of motion for the WILL-membrane subsystem are of the same form as Eqs. (50)–(52). The Einstein-Maxwell equations of motion read

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 8\pi G_N (T_{\mu\nu}^{(\text{EM})} + T_{\mu\nu}^{(\text{brane})}), \qquad (71)$$

$$\partial_\nu (\sqrt{-G} G^{\mu\kappa} G^{\nu\lambda} \mathcal{F}_{\kappa\lambda}) + j^\mu = 0, \qquad (72)$$

where

$$T_{\mu\nu}^{(\text{EM})} \equiv \mathcal{F}_{\mu\kappa} \mathcal{F}_{\nu\lambda} G^{\kappa\lambda} - G_{\mu\nu} \frac{1}{4} \mathcal{F}_{\rho\kappa} \mathcal{F}_{\sigma\lambda} G^{\rho\sigma} G^{\kappa\lambda}, \quad (73)$$

$$T_{\mu\nu}^{(\text{brane})} \equiv -G_{\mu\kappa} G_{\nu\lambda} \int d^3\sigma \frac{\delta^{(4)}(x - X(\sigma))}{\sqrt{-G}}$$
$$\times \chi \sqrt{-\gamma} \gamma^{ab} \partial_a X^\kappa \partial_b X^\lambda, \qquad (74)$$

$$j^\mu \equiv q \int d^3\sigma \delta^{(4)}(x - X(\sigma)) \varepsilon^{abc} F_{bc} \partial_a X^\mu. \qquad (75)$$

We find the following self-consistent spherically-symmetric stationary solution for the coupled Einstein-Maxwell-WILL-membrane system (70). For the Einstein subsystem we have a solution:

$$(ds)^2 = -A(r)(dt)^2 + A^{-1}(r)(dr)^2$$
$$+ r^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2], \qquad (76)$$

consisting of two different black holes with a *common* event horizon:

(a) Schwarzschild black hole inside the horizon:

$$A(r) \equiv A_-(r) = 1 - \frac{2GM_1}{r},$$
$$\text{for} \quad r < r_0 \equiv r_{\text{horizon}} = 2GM_1. \qquad (77)$$

(b) Reissner-Norström black hole outside the horizon:

$$A(r) \equiv A_+(r) = 1 - \frac{2GM_2}{r} + \frac{GQ^2}{r^2}, \qquad (78)$$

for $r > r_0 \equiv r_{\text{horizon}}$,

where $Q^2 = 8\pi q^2 r_{\text{horizon}}^4 \equiv 128\pi q^2 G^4 M_1^4$; For the Maxwell subsystem we have $\mathcal{A}_1 = \ldots = \mathcal{A}_{D-1} = 0$ everywhere and

(c) Coulomb field outside horizon:

$$\mathcal{A}_0 = \frac{\sqrt{2} q r_{\text{horizon}}^2}{r}, \quad \text{for} \quad r \geq r_0 \equiv r_{\text{horizon}}. \qquad (79)$$

(d) No electric field inside horizon:

$$\mathcal{A}_0 = \sqrt{2} q r_{\text{horizon}} = \text{const.}, \qquad (80)$$

for $r \leq r_0 \equiv r_{\text{horizon}}$.

Using the same (synchronous) gauge choice (38) and ansatz for the dual "gauge potential" (39), as well as taking into account (79) and (80), the WILL-membrane equations of motion (50)–(52) acquire the form (recall $(\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$):

$$(\partial_0 X \partial_0 X) = 0, \qquad (\partial_0 X \partial_i X) = 0, \qquad (81)$$

$$(\partial_i X \partial_j X) - \frac{1}{2} \gamma_{ij} \gamma^{kl} (\partial_k X \partial_l X) = 0, \qquad (82)$$

(these constraints are the same as in the absence of cou-

pling to space-time gauge field (41) and (42));

$$\partial_0(\sqrt{\gamma_{(2)}}\gamma^{kl}(\partial_k X \partial_l X)) = 0, \qquad (83)$$

(once again the same equation as in the absence of coupling to space-time gauge field (43));

$$\tilde{\Box}^{(3)} X^\mu + (-\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{kl}\partial_k X^\nu \partial_l X^\lambda)\Gamma^\mu_{\nu\lambda}$$
$$- q\frac{\gamma^{kl}(\partial_k X \partial_l X)}{\sqrt{2}\chi}\partial_0 X^\nu(\partial_r \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\lambda)G^{\lambda\mu} = 0. \quad (84)$$

Here $\chi \equiv T_0 - \sqrt{2}q\mathcal{A}_0$ with $\mathcal{A}_0$ as in (79) and (80) is the variable brane tension coming from Eqs. (39) and (48); $X^0 \equiv t, X^1 \equiv r, X^2 \equiv \theta, X^3 \equiv \phi$; and

$$\tilde{\Box}^{(3)} \equiv -\frac{1}{\chi\sqrt{\gamma^{(2)}}}\partial_0(\chi\sqrt{\gamma^{(2)}}\partial_0)$$
$$+ \frac{1}{\chi\sqrt{\gamma^{(2)}}}\partial_i(\chi\sqrt{\gamma^{(2)}}\gamma^{ij}\partial_j). \qquad (85)$$

A self-consistent solution to Eqs. (81)–(84) reads

$$X^0 \equiv t = \tau, \qquad \theta = \sigma^1, \qquad \phi = \sigma^2, \qquad (86)$$

$$r(\tau, \sigma^1, \sigma^2) = r_{\text{horizon}} = \text{const.}, \quad A_\pm(r_{\text{horizon}}) = 0, \quad (87)$$

i.e., the WILL membrane automatically positions itself on the common event horizon of the pertinent black holes. Furthermore, inserting (86) and (87) in the expression (74) for the WILL-membrane energy-momentum tensor $T^{(\text{brane})}_{\mu\nu}$ and using the simple expressions for the components of the Ricci tensor corresponding to the metric (76) $R^0_0 = R^1_1 = -\frac{1}{2r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}A(r))$ [13], the Einstein Eqs. (71) entail the following important matching conditions for the space-time metric components (76) along the WILL-membrane surface:

$$\frac{\partial}{\partial r}A_+\bigg|_{r=r_{\text{horizon}}} - \frac{\partial}{\partial r}A_-\bigg|_{r=r_{\text{horizon}}} = -16\pi G\chi. \quad (88)$$

The matching condition (88) corresponds to the so called statically soldering conditions in the theory of lightlike thin shell dynamics in general relativity in the case of "horizon straddling" lightlike matter (first Ref. [4]). Here, condition (88) yields relations between the parameters of the black holes and the WILL membrane (q being its surface charge density):

$$M_2 = M_1 + 32\pi q^2 G^3 M_1^3 \qquad (89)$$

and for the brane tension χ:

$$\chi \equiv T_0 - 2q^2 r_{\text{horizon}} = q^2 G M_1, \quad \text{i.e.} \quad T_0 = 5q^2 G M_1. \qquad (90)$$

We would like to stress that the present WILL-brane models provide a systematic dynamical description of lightlike branes (as sources for both gravity and electro-magnetism) from first principles starting with concise Weyl-conformally invariant actions (46) and (70). It is interesting that out of the several possibilities discussed in the first Ref. [4] for lightlike matter moving in a black hole gravitational field only the "horizon straddling" is selected by the WILL branes.

VII. WILL-MEMBRANE DYNAMICS IN KALUZA-KLEIN PRODUCT SPACES

Here we consider WILL membrane moving in a general product-space $D = (d + 2)$-dimensional gravitational background $\mathcal{M}^d \times \Sigma^2$ with coordinates (x^μ, y^m) ($\mu = 0, 1, \ldots, d - 1, m = 1, 2$). First we take the following simple form for the Riemannian metric (which is applicable also to brane-world scenarios):

$$(ds)^2 = f(y)g_{\mu\nu}(x)dx^\mu dx^\nu + h_{mn}(y)dy^m dy^n. \qquad (91)$$

The metric $g_{\mu\nu}(x)$ on $\mathcal{M}^d$ is of Lorentzian signature. Furthermore, we assume that the WILL brane wraps around the "internal" space Σ^2 and choose the following ansatz (recall $\tau \equiv \sigma^0$) which uses the identity mapping between the brane coordinates σ^1, σ^2 and the coordinates Y^m of the brane:

$$X^\mu = X^\mu(\tau), \qquad Y^m = \sigma^m,$$
$$\gamma_{mn} = a(\tau)h_{mn}(\sigma^1, \sigma^2). \qquad (92)$$

Then the equations of motion and constraints (41)–(45) reduce to

$$\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu}(X) = 0,$$
$$\frac{1}{a(\tau)}\partial_\tau(a(\tau)\partial_\tau X^\mu) + \partial_\tau X^\nu \partial_\tau X^\lambda \Gamma^\mu_{\nu\lambda}(g) = 0. \qquad (93)$$

Here $a(\tau)$ is the conformal factor of the spacelike part of the internal membrane metric (last Eq. (92)) and $\Gamma^\mu_{\nu\lambda}(g)$ is the Christoffel connection for the noncompact space metric $g_{\mu\nu}(x)$. Also, let us note the that the overall conformal factor $f(u)$ of the metric on $\mathcal{M}^d$ (91) drops out completely.

Eqs. (93) are of the same form as the equations of motion for a massless point particle with a worldline "einbein" $e = a^{-1}$ moving in $\mathcal{M}^d$. In other words, in this situation we deal with a membrane living in the extra internal dimensions Σ^2 and moving as a whole with the speed of light in "ordinary" space-time $\mathcal{M}^d$—its motion is indistinguishable from the dynamics of a regular massless point particle w.r.t. the noncompact projected world $\mathcal{M}^d$. Notice, however, that although being massless, the particlelike brane mode acquires nontrivial Kaluza-Klein quantum numbers due to the WILL-brane winding of the extra compact dimensions—a very peculiar situation in the context of Kaluza-Klein theories.

Now, we take more complicated form for the product-space Riemannian metric:

$$(ds)^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + v(x)h_{mn}(y)dy^m dy^n, \qquad (94)$$

where we allow for a variable size (squared) $v(x)$ of the internal compact dimensions. Employing the ansatz:

$$X^\mu = X^\mu(\tau, \sigma^1, \sigma^2), \qquad Y^m = \sigma^m,$$

$$\gamma_{mn} = a(\tau)h_{mn}(\sigma^1, \sigma^2) \tag{95}$$

the constraints and equations of motion (41)–(45) acquire the following form:

$$(\partial_0 X \partial_0 X) = 0, \qquad (\partial_0 X \partial_i X) = 0,$$

$$(\partial_i X \partial_j X) - \frac{1}{2}h_{ij}h^{kl}(\partial_k X \partial_l X) = 0, \tag{96}$$

$$\hat{\Box}^{(3)}X^\mu + \left(-\partial_0 X^\nu \partial_0 X^\lambda + \frac{1}{a}h^{kl}\partial_k X^\nu \partial_l X^\lambda\right)\Gamma^\mu_{\nu\lambda}(g)$$

$$-\frac{1}{a}g^{\mu\nu}\frac{\partial v}{\partial x^\nu}\bigg|_{x=X} = 0, \tag{97}$$

with

$$\hat{\Box}^{(3)} \equiv -\frac{1}{a}\partial_0(a\partial_0) + \frac{1}{\sqrt{h}}\partial_i(\sqrt{h}h^{ij}\partial_j); \tag{98}$$

$$\partial_i X^\mu \frac{\partial v}{\partial x^\mu}\bigg|_{x=X} = 0. \tag{99}$$

Let us also note that Eq. (43) (the remnant of the equations of motion for the dual gauge potential u being a consequence of the constraints and the rest of the equations of motion) upon using the ansatz (95) assumes the form:

$$\partial_0(h^{ij}(\partial_i X \partial_j X) + 2v(X)) = 0. \tag{100}$$

In what follows we will study the particlelike mode dynamics of the WILL membrane, i.e., we will use the ansatz (92). Then Eqs. (96)–(99) reduce to

$$\partial_0 X^\mu g_{\mu\nu}(X)\partial_0 X^\nu = 0, \tag{101}$$

$$\frac{1}{a}\partial_0(a\partial_0 X^\mu) + \partial_0 X^\nu \partial_0 X^\lambda \Gamma^\mu_{\nu\lambda}(g) + \frac{1}{a}g^{\mu\nu}\frac{\partial v}{\partial x^\nu}\bigg|_{x=X} = 0. \tag{102}$$

Let us particularly stress that Eqs. (101) and (102) describe *massless* particlelike dynamics in a "potential" $v(X)$ (the space-dependent size-squared of the extra compact dimensions) which is an essential new feature stemming from the WILL-membrane model. These equations *cannot* be derived from a reparametrization-invariant (massless) point-particle-like action.

For a static spherically-symmetric (w.r.t. noncompact dimensions) background:

$$(ds)^2 = -A(r)(dt)^2 + B(r)(dr)^2 + C(r)[(d\theta)^2 + \sin^2(\theta)$$

$$\times (d\phi)^2] + v(r)h_{mn}(y)dy^m dy^n, \tag{103}$$

and identifying as usual $X^0 \equiv t = \tau$ we obtain from (101) and (102):

(a) As a consequence of Eqs. (101) and (102) we have

$$\partial_0 X^\mu \frac{\partial v}{\partial x^\mu}\bigg|_{x=X} = 0, \tag{104}$$

which yields $r = r_0 \equiv$ const. (τ independent).

(b) The equation of motion (102) for $\mu = 0$ yields $a = a_0 \equiv$ const. (τ independent). Recall from last Eq. (92) that a has the meaning of a size squared of the world surface (at fixed proper time) of the WILL brane w.r.t. internal world-volume metric γ_{ab}.

(c) Taking the above relations into account, the equation of motion (102) for $\mu = r$ yields a purely functional equation:

$$\left(\frac{1}{a_0}\frac{\partial v}{\partial r} - \frac{1}{2}A\frac{\partial}{\partial r}\ln\frac{C}{A}\right)\bigg|_{r=r_0} = 0, \tag{105}$$

which determines a set of allowed constant values for $r = r_0 \equiv r(a_0)$ depending on the explicit form of the background (103) and parametrically depending on a_0.

(d) Finally, the lightlike constraint (first Eq. (101)) and the equations of motion (102) for the spacelike noncompact coordinates $\underline{X} \equiv (X^1, X^2, X^3)$ (recall $r \equiv \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2} = r_0$) acquire the form:

$$\dot{\underline{X}}^2 = \frac{A}{C}r^2\bigg|_{r=r_0}, \qquad \ddot{\underline{X}} + \frac{A}{C}\bigg|_{r=r_0}\underline{X} = 0. \tag{106}$$

Here and below the scalar products of 3 vectors are understood w.r.t. flat metric.

Obviously, the solution to (106) are as follows:

$$\underline{X} = r_0(\underline{n}^{(1)}\cos(\omega\tau) + \underline{n}^{(2)}\sin(\omega\tau)), \qquad \omega^2 \equiv \frac{A}{C}\bigg|_{r=r_0}, \tag{107}$$

where r_0 is determined from Eq. (105) and $\underline{n}^{(1,2)}$ are two constant mutually orthogonal unit 3 vectors: $(\underline{n}^{(1)})^2 = (\underline{n}^{(2)})^2 = 1$, $\underline{n}^{(1)}.\underline{n}^{(2)} = 0$. In other words, the projection of the WILL membrane on the noncompact space-time rotates with the speed of light in a two-dimensional plane defined by the unit vectors $\underline{n}^{(1,2)}$, traversing a circle with radius r_0 determined from Eq. (105) with angular velocity ω given by the second relation (107).

Solution (107) describes nontrivial massless mode dynamics with energy E and angular momentum $|\underline{M}|$ (recall $\chi = T_0$ (40)):

$$E = T_0 a_0 A(r_0)\Omega, \qquad |\underline{M}| = T_0 a_0\sqrt{A(r_0)C(r_0)}\Omega, \tag{108}$$

Ω being the volume of the compact internal space, which implies the relation:

$$E = |\underline{M}|\omega(r_0). \tag{109}$$

(recall $\omega(r_0) \equiv \sqrt{A(r_0)/C(r_0)}$) (107)). Equations (108) follow straightforwardly from the expressions for the Noether conserved currents corresponding to invariance of the action (33) with external metric background given by (103) under translation of $X^0 \equiv t$ and ϕ (external space-time spherical angle).

Following Refs. [14,15], we have the following explicit solution of the Einstein equations in the case of two extra dimensions for the metric (103):

$$A(r) = \left(\frac{\alpha r - 1}{\alpha r + 1}\right)^{2p+q},$$

$$C(r) = r^2 B(r) = \frac{1}{\alpha^4 r^2} \frac{(\alpha r + 1)^{2(p+1)-q}}{(\alpha r - 1)^{2(p-1)-q}}, \quad v(r) \quad (110)$$

$$= \left(\frac{\alpha r + 1}{\alpha r - 1}\right)^q,$$

whereas the compact internal space Σ^2 is a torus with $h_{mn}(y) = \delta_{mn}$ (therefore, the world surface of the WILL membrane is similarly assumed to have toroidal topology). In (110) α is arbitrary positive integration constant of mass dimension 1, p and q are free numerical parameters subject to the relation $p^2 + \frac{1}{2}q^2 = 1$, and the resulting metric is well defined in the region $\alpha r > 1$.

Inserting the expressions (110) into Eq. (105) we obtain

$$0 = \frac{4q}{a_0} \frac{(\alpha r_0 + 1)^{q-1}}{(\alpha r_0 - 1)^{q+1}} + \left(\frac{\alpha r_0 - 1}{\alpha r_0 + 1}\right)^{p-q} \left[\frac{(4p + 2)}{\alpha r_0 + 1}\right.$$

$$\left. - \frac{2}{\alpha r_0} - \frac{(4p - 2)}{\alpha r_0 - 1}\right], \quad (111)$$

determining allowed values of the radius $r_0 = r_0(a_0)$ of the planar circular orbits as a function of a_0 (recall that a_0 is strictly positive free parameter—the size squared of the WILL membrane w.r.t. its internal world-volume metric, cf. Eq. (92)). From Eq. (111) and taking into account the constraints on the parameters in (110), we have:

(a) For $0 < q \leq 1/\sqrt{2}$ and $\frac{1}{2} \leq p \leq 1 - \frac{1}{2}q^2$ the allowed values of the radius $r_0(a_0)$ of the planar circular orbits lie in the interval $(\frac{1}{\alpha}, \frac{2}{\alpha} \times (p + \sqrt{p^2 - 1/4})$.

(b) For $-1/\sqrt{2} \leq q < 0$ and $\frac{1}{2} \leq p \leq 1 - \frac{1}{2}q^2$ the allowed values of $r_0(a_0)$ are $r_0(a_0) \geq \frac{2}{\alpha} \times (p + \sqrt{p^2 - 1/4})$; we have in this case $r_0(a_0) \longrightarrow \infty$ for $a_0 \to 0$, and $r_0(a_0) \longrightarrow \frac{2}{\alpha}(p + \sqrt{p^2 - 1/4})$ for $a_0 \to \infty$.

Let us note that in the case $q > 0$, (110) also yields the interior solution for the "gravitational bags" [16] (the latter similarly require $p > \frac{1}{2})^2$.

For both situations above one can easily show that the allowed values for the angular velocity (cf. the definition in

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^{2}For "gravitational bags" [16] the exterior solution, matched to the interior one through a regular domain wall, is a Schwarzschild metric.

Eq. (107)):

$$\omega(r_0) = \alpha^2 r_0 \frac{(\alpha r_0 - 1)^{2p-1}}{(\alpha r_0 + 1)^{2p+1}} \quad (112)$$

lie in a finite interval between 0 and $\omega_{\max}$:

$$\omega_{\max} = \frac{\alpha}{2} \frac{(p + \sqrt{p^2 - 1/4})(p - 1/2 + \sqrt{p^2 - 1/4})^{2p-1}}{(p + 1/2 + \sqrt{p^2 - 1/4})^{2p+1}}. \quad (113)$$

Let us note that in the case $v = \text{const.}$ (constant size of extra dimensions) the free scale parameter a_0 disappears in Eq. (105) leading to a qualitatively different situation. For instance, in the case of Schwarzschild metric on the noncompact space, i.e., $A(r) = B^{-1}(r) = 1 - 2GM/r$, $C(r) = r^2$, $v = \text{const.}$ in (103), Eq. (105) is satisfied for only two special values of r [17]: $r_0 = 2GM$ (massless particle "sitting" on the horizon) and $r_0 = 3GM$ (massless particle on an unstable circular orbit). On the other hand, for variable size of the extra dimensions ($\frac{\partial v}{\partial r} \neq 0$) a continuous range of values for the "radius" of the planar circular orbits is available corresponding to the solutions $r = r_0(a_0)$ of Eq. (105).

Let us particularly emphasize the fact that, although the WILL brane is wrapping the extra (compact) dimensions in a topologically nontrivial way (cf. second Eq. (95)), its modes remain *massless* from the projected noncompact space-time point of view. This is a new phenomenon from the point of view of Kaluza-Klein theories: here we have particlelike membrane modes, which acquire nonzero quantum numbers due to nontrivial winding, while at the same time these particlelike modes remain massless. In contrast, one should recall that in ordinary Kaluza-Klein theory (for reviews, see [18]), nontrivial dependence on the extra dimensions is possible for point particles or even standard strings and branes only at a very high-energy cost (either by momentum modes or winding modes), which implies a very high mass from the projected noncompact space-time point of view.

VIII. CONCLUSIONS AND OUTLOOK

In the present paper we have discussed in detail a completely new type of p branes. The use of a modified non-Riemannian volume form (integration measure) in their Lagrangian actions was of crucial importance. Next, formulating acceptable p-brane dynamics naturally requires the introduction of additional world-volume gauge fields. Employing a square-root Maxwell-type action for auxiliary world-volume gauge field was most instrumental for achieving a consistent p-brane theory which is manifestly Weyl-conformally invariant for any p and, furthermore, for any even p (odd-dimensional world volume) it describes intrinsically lightlike p branes. Remarkably, the brane tension becomes now a gauge-dependent concept—

it appears as a composite field transforming nontrivially under Weyl-conformal transformations. Unlike previous Weyl-invariant *reformulations* of the standard Weyl non-invariant Nambu-Goto p branes (which preserve the physical content of the Nambu-Goto branes and, therefore, describe massive brane modes), the presently discussed new class of Weyl-invariant p branes for $p + 1 =$ odd describes genuine massless lightlike branes.

Weyl-invariant lightlike p branes (WILL branes) offer a broad variety of interesting physical applications, most notably in the context of black hole physics and Kaluza-Klein theories. In the case of a WILL membrane moving as a test brane in a gravitational black hole background we have seen that it positions itself automatically on the event horizon. Furthermore, we studied a self-consistent solution of the coupled Einstein-Maxwell-WILL-membrane system where the WILL membrane appears as a source for both gravity and electromagnetism. This self-consistent solution has the WILL membrane sitting at (''straddling'') the common event horizon of a Schwarzschild (in the interior) and Reissner-Nordström (in the exterior) black hole solutions. This is an indication that the WILL-membrane model indeed provides a plausible explicit dynamical realization of the so called ''membrane paradigm'' in black hole physics [2]. The quantization of the WILL-membrane dynamics under these circumstances may be very much related to the quantization of the horizon degrees of freedom. Indeed, the WILL membrane presents a remarkable resemblance to the stringlike objects introduced by 't Hooft [19] to characterize the horizon degrees of freedom.

In the context of Kaluza-Klein theories the WILL branes appear also to play a very interesting role. Indeed, we have found solutions for the WILL membrane moving in higher-dimensional Kaluza-Klein-type space-times which describe the dynamics of *massless* particlelike brane modes even though the membrane itself is wrapping the extra compact dimensions and, therefore, acquires nontrivial Kaluza-Klein charges—a situation inaccessible in the context of standard Kaluza-Klein theories. When the size of the extra compact dimensions has a nontrivial space dependence like in some self-consistent solutions of higher-dimensional Einstein equations [14,15], the behavior of the massless particlelike brane mode solutions is quite interesting from the point of view of the noncompact $D = 4$ space-time point of view. These massless brane modes are trapped on finite planar circular orbits with linear dependence between energy and angular momentum. The pa-

rameters of the metric on the noncompact part of the Kaluza-Klein space-time dictate that the allowed values of the angular velocity lie in a finite interval.

It is essential to note that the above massless particle-like dynamics is a special feature due to its WILL brane (31) origin. It cannot be derived neither from a reparametrization-invariant point-particle action nor by zero-mode reduction of a Nambu-Goto-type brane action.

There are various physically interesting directions for further systematic study of the properties and implications of the new class of Weyl-conformally invariant branes discussed above, such as: quantization (Weyl-conformal anomaly and critical dimensions); supersymmetric generalization; possible relevance for the open string dynamics (similar to the role played by Dirichlet- (Dp-)branes); WILL-brane dynamics in more complicated gravitational black hole backgrounds (e.g., Kerr-Newman); WILL-brane dynamics in more complicated Kaluza-Klein-type space-times, including more complex winding of the extra dimensions. To this end let us note that there exist physically interesting solutions of higher-dimensional Einstein equations—''gravitational bags'' [16] and ''dimension bubble'' solutions [20], where the presence of a domain wall implies big gradient for the size squared $v(r)$ of the extra dimensions (cf. Eq. (105)). Thus, it would be very interesting to study WILL-brane dynamics in such Kaluza-Klein backgrounds.

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